THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH3070 Introduction to Topology 2017-2018
Suggested Solution for Quiz 2

1. (Pictures omitted)

   (a) The subspace topology of A is homeomorphic to the upper/lower limit topology of $\mathbb{R}$.

   (b) The subspace topology of A is homeomorphic to the discrete topology of $\mathbb{R}$.

2. (a) The equivalence class of the point $(1/2, 0$ is given by $\{(x_1, x_2) \mid 4x_1^2 + 9x_2^2 = 1\}$. It is an ellipse.

   (b) Consider the preimage of the complement of A under the quotient map. One can show that we have
   $$\pi^{-1}(A^c) = \{(x_1, x_2) \in \mathbb{R}^2 \mid 0 \leq 4x_1^2 + 9x_2^2 < 4 \text{ or } 4x_1^2 + 9x_2^2 > 25\},$$
   which is open in $\mathbb{R}^2$. Hence A^c is open and A is closed.

   (c) Let $F : (\mathbb{R}^2/\sim, \mathfrak{T}_{\text{quot}}) \to ([0, \infty), \mathfrak{T}_{std})$ be the mapping defined by $F([(x_1, x_2)]) = 4x_1^2 + 9x_2^2$.
   First of all, note that for each representative of the equivalence classes, by definition the value $4x_1^2 + 9x_2^2$ are the same. So the mapping F is well-defined.
   Besides, for any $c \in [0, \infty)$, we have $F([\frac{\sqrt{c}}{2}, 0]) = c$. Hence F is surjective.
   Furthermore, if $F([x_1, x_2]) = F([y_1, y_2])$, we will have $4x_1^2 + 9x_2^2 = 4y_1^2 + 9y_2^2$. This implies that $(x_1, x_2) \sim (y_1, y_2)$.
   As a result, F is bijective. To show that F is continuous, consider the following commutative diagram:
   $$\begin{tikzcd} \mathbb{R}^2 \arrow[d, "\pi"] \arrow[dr, "F \circ \pi"] & \\ \mathbb{R}^2/\sim \arrow[r, "F"] & {[0, \infty)} \end{tikzcd}$$
   Note that the function $F \circ \pi : \mathbb{R}^2 \to [0, \infty)$ is given by $F \circ \pi(x_1, x_2) = 4x_1^2 + 9x_2^2$, which is continuous. Therefore, by the properties of quotient topology, F is continuous.

3. (a) $\mathbb{R}_{ll}$ is Hausdorff. Given any two distinct points $x, y \in \mathbb{R}_{ll}$, let $d = |x - y|$. Then $x \in [x, x + d)$, $y \in [y, y + d)$ and $[x, x + d) \cap [y, y + d) = \emptyset$.

   (b) $\mathbb{R}_{ll}$ is normal. Given and two disjoint closed sets A and B. Note that $A \in \mathbb{R}\backslash B$. Since B is closed, $\mathbb{R}\backslash B$ is open. In particular, for any $a \in A$, there exists $a' \in \mathbb{R}$ such that $a \in [a, a') \subset \mathbb{R}\backslash B$. Let $U = \cup_{a \in A}[a, a')$. Then we have $A \subset U \subset \mathbb{R}\backslash B$. Similarly, we may construct an open set $V = \cup_{b \in B}[b, b')$ such that $B \subset V \subset \mathbb{R}\backslash A$. We are going to show that $U \cap V = \emptyset$.

   Suppose we have some $c \in U \cap V$. Then there exists some a, a', b, b' such that $c \in [a, a') \cap [b, b')$. In particular, we have (i) $b \in [a, a')$ or (ii) $a \in [b, b')$. WLOG assume (i) is true. Then we have $b \in U \subset \mathbb{R}\backslash B$, contradiction. Hence $U \cap V = \emptyset$.

(c) $\mathbb{R}_{ll}$ is not compact. Consider the open cover $\{I_n = (-n, n)\}_{n\in\mathbb{N}}$. Suppose a finite subcover $\{I_{n_1}, I_{n_2}, \ldots, I_{n_k}\}$ exists. Let $n_{max} = \max\{n_1, n_2, \ldots, n_k\}$. Then we have $\mathbb{R} = (-n_{max}, n_{max})$. In particular $(n_{max} + 1) \notin \mathbb{R}$, contradiction.

4. (a) Consider an arbitrary base element U containing 0. Then there exists $M \in \mathbb{N}$ such that

$$U = U_1 \times U_2 \times \cdots \times U_M \times \mathbb{R} \times \mathbb{R} \times \ldots$$

In particular, we have $0 \in U_k$ for any $k = 1, 2, \ldots M$. Therefore, for any $n \geq M$, since $x_n(k) = 0 \in U_k$ for any $k = 1, 2, \ldots, M$, we have $x_n \in U$. Hence $x_n \to 0$ in the product topology.

(b) x_n does not converges to 0 in box topology. Consider the open set $V = \prod_{n\in\mathbb{N}}(-0.5, 0.5)$. Clearly $0 \in V$. However, since for any $n \in \mathbb{N}$, we have $x_n(n+1) = 1 \notin (-0.5, 0.5)$. Hence $x_n \notin V$ for all $n \in \mathbb{N}$.

5. (a) Let $\{U_\alpha\}_{\alpha\in I}$ be an open cover. Then there exists some U_0 such that $x_0 \in U_0$. Since $x_n \to x$, there exists $N \in \mathbb{N}$ such that for any $n \geq N$, we have $x_n \in U_0$. Pick U_i such that $x_i \in U_i$ for $i = 1, 2, \ldots, N-1$. Then $\{U_i\}_{i=1}^n$ is a finite subcover.

(b) Since A is discrete, $\{\{a\}\}_{a\in A}$ is an open cover of A. Since A is compact, there exists $a_1, a_2, \ldots, a_n$ such that $A = \cup_{i=1}^n \{\{a_i\}\} = \{a_1, a_2, \ldots, a_n\}$.

(c) Given any open cover $\mathfrak{C} = \{C_i\}_{i\in I}$ of X. For any $i \in I$, we have $C_i = \cup_j B_j^{(i)}$. In particular, $\{B_j^{(i)}\}_{i,j} \subset \mathfrak{B}$ is an open cover of X. By assumption, we have a finite subcover $\{B_{j_k}^{(i_k)}\}$. Since $B_{j_k}^{(i_k)} \subset C_{i_k}$, $\{C_{i_k}\}$ is a finite subcover.

(d) For any $x \in X$, by assumption $X\backslash\{x\}$ is compact. Since X is Hausdorff, $X\backslash\{x\}$ is closed. Hence $\{x\}$ is open. This implies that $(X, \mathfrak{T})$ is discrete.

6. (a) Let F be a closed set. Since X is compact, F is also compact. Since ϕ is continuous, $\phi(F)$ is also compact. Since Y is Hausdorff, $\phi(F) \subset Y$ is closed.

(b) We are going to show that $\pi_2(H)^c$ is open. Pick any $y \in \pi_2(H)^c$. Then for any $x \in X$, we have $(x, y) \notin H$. Since $H \subset X \times Y$ is closed, for each x there exists $U_x \in \mathfrak{T}_X$ and $V_x \in \mathfrak{T}_Y$ such that $(x, y) \in U_x \times V_x \subset H^c$. In particular, $\{U_x\}_{x\in X}$ is an open cover of X. Since X is compact, there exists a finite subcover $U_{x_1}, U_{x_2}, \ldots, U_{x_n}$. Let $V = \cap_{i=1}^n V_{x_i}$. Note that $y \in V$ and V is open.

Next we are going to show that $V \cap \pi_2(H) = \emptyset$. For any $y' \in V$, if $y' \in \pi_2(H)$, then there exists some $x' \in X$ such that $(x', y') \in H$. Since $X = \cup_{i=1}^n U_{x_i}$, we have $x' \in U_{x_k}$ for some k. Since $y' \in V = \cap_{i=1}^n V_{x_i}$, $y' \in V_k$. Hence $(x', y') \in U_{x_k} \times V_{x_k}$. However, by definition of U_{x_k} and V_{x_k}, we have $(U_{x_k} \times V_{x_k}) \cap H = \emptyset$, contradiction.

As a result, we have $y \in V \subset \pi_2(H)^c$. Hence $\pi_2(H)$ is closed.